Extensive Scaling and Nonuniformity of the Karhunen-Loève Decomposition for the Spiral-Defect Chaos State

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(February 5, 2008)

Abstract

By analyzing large-aspect-ratio spiral-defect-chaos (SDC) convection images, we show that the Karhunen-Loève decomposition (KLD) scales extensively for subsystem-sizes larger than $4d$ (d is the fluid depth), which strongly suggests that SDC is extensively chaotic. From this extensive scaling, the intensive length $\xi_{\rm KLD}$ is computed and found to have a different dependence on the Rayleigh number than the two-point correlation length ξ_2. Local computations of $\xi_{\rm KLD}$ reveal a substantial spatial nonuniformity of SDC that extends over radii $18d < r < 45d$ in a $\Gamma = 109$ aspect-ratio cell.

47.27.Cn, 05.70.Ln, 05.45.+b,

A significant theoretical challenge is to find ways to characterize the nonperiodic time-dependent patterns often observed in large sustained nonequilibrium systems [1]. In a recent paper [2], a new length scale for characterizing spatiotemporal chaos (STC), the KLD length ξ_{KLD}, was proposed based on the *extensive scaling* of the Karhunen-Loève decomposition (KLD) [3]. This length scale was shown [2] to be computable from moderate amounts of space-time data and to contain information similar to the fractal-dimension density, which has not yet been computed from experimental data [4]. For certain idealized mathematical models, the length ξ_{KLD} was shown to have a different parametric dependence than the commonly computed two-point correlation length ξ_2 and so provides a different way to characterize spatiotemporal chaos [2]. Further, this length ξ_{KLD} could be calculated from data localized to a region of space and so offered a way to analyze spatial inhomogeneities. However, an application of the KLD length to *experimental* data had not been made prior to this work.

In this paper, we present the first application of the KLD length to experimental STC data by analyzing the recently discovered [5] spiral-defect chaos (SDC) in Rayleigh-Bénard convection. We analyze the SDC state of large aspect-ratio ($\Gamma \equiv r/d = 29$ and 109 where r and d are the radius and thickness of the cell) cells and find that the KLD dimension D_{KLD} of the data scales extensively with subsystem volume when the diameter of the subvolume is larger than about $4d$ where d is the depth of the fluid. This extensivity strongly suggests that the SDC state is extensively chaotic [4, 6], which provides the first such evidence for an experimental system. From the extensive scaling of the dimension D_{KLD} with subvolume, we calculate the length ξ_{KLD} which reflects the density of linearly independent modes needed to approximate the spatiotemporal data [2]. For the reduced Rayleigh numbers ϵ explored in this study, ξ_{KLD} exhibits a minimum with increasing ϵ while the two-point correlation length ξ_2 monotonically decreases. We speculate that this different behavior in the KLD length arises from changes in the structure of spirals and straight rolls with increasing ϵ. Finally, we also show that the length ξ_{KLD} provides a way to measure a dynamical inhomogeneity in SDC data within the interior of the cell. A local KLD analysis demonstrates that SDC is *not* dynamically uniform (Most numerical simulations of SDC [7] have used periodic lateral boundary conditions, for which the dynamics should be statistically homogeneous by translational invariance). In addition, this nonuniformity extends to within a radius $r = 18d$ of the center of an aspect-ratio $\Gamma = 109$ cell. These results suggest that the length ξ_{KLD} can provide useful insights about spatiotemporal chaos.

We tested the utility of the length ξ_{KLD} for analyzing SDC experimental data [5, 8] by collecting thousands of shadowgraph images [9] of the SDC state from a Rayleigh-Bénard experiment using compressed CO_2 gases with Prandtl number $\sigma \approx 1$ and in cylindrical cells of aspect ratio $\Gamma = 29$ and 109. SDC appeared for $\epsilon \equiv \Delta T/\Delta T_c - 1$ above 0.56 and 0.23 respectively for the $\Gamma = 29$ and 109 cells, where ΔT_c is the critical temperature difference for the onset of convection. Five sets of SDC data were taken in the two cells as summarized in Table I. We chose sampling times Δt that were significantly longer than the correlation time $\tau \sim 10 t_v$ (where t_v is the vertical thermal diffusion time) to maximize the amount of uncorrelated data [5]. Good spatial resolution of the convection patterns required at least 5 pixels per convection roll for the $\Gamma = 109$ cell. Data was taken from the central 43% of the $\Gamma = 109$ convection cell, as indicated in Fig. 1.

To illustrate the KLD analysis used to study the experimental data, let $u(t_i, \mathbf{x}_j)$ denote

the light intensity of an experimental image at position $\mathbf{x}_j$ at time t_i. The analysis proceeds by constructing a $T \times S$ space-time matrix of data,

$$A_{ij} \;=\; u(t_i, \mathbf{x}_j) - < u(t_i, \mathbf{x}_j) > \tag{1}$$

where $< u(t_i, \mathbf{x}_j) >$ denotes the time-average (average over index i) of the set of experimental images $u(t_i, \mathbf{x}_j)$, T is the number of observation times t_i, and S is the number of observation sites $\mathbf{x}_j$. The KLD dimension D_{KLD} [10] of the matrix A_{ij} then measures the number of linear eigenmodes needed to approximate some fraction $0 < f < 1$ of the variance of the experimental data and can be computed from the eigenvalues of the matrix $\mathbf{A}^T\mathbf{A}$ [2, 10, 11]. We compute $D_{\mathrm{KLD}}(\mathbf{x_j})$ for concentric subsystems of volume V (square or circular geometry) that are centered at a particular point $\mathbf{x_j}$ in space. $D_{\mathrm{KLD}}(\mathbf{x_j})$ depends on the point $\mathbf{x_j}$ and so provides a measure of dynamical inhomogeneity. If $D_{\mathrm{KLD}}(\mathbf{x_j})$ increases linearly with subsystem volume V and with a slope δ, then the length ξ_{KLD} is defined to be $\delta^{-1/\mathrm{d}}$ where d is the dimensionality of the data (d = 2 for SDC).

In applying these ideas to SDC data, we computed D_{KLD} for a fixed fraction $f = 0.7$ [12] and for larger and larger square subimages in the center of the convection cell of size $S \times S$, where $2d < S < 25d$. As shown in Fig. 2, the dimension D_{KLD} scales approximately linearly with subsystem data over a relatively large range of subsystem sizes $4d < S < 13d$ provided that a sufficiently long time series was used. This extensive scaling, together with the arguments in Ref [2] relating ξ_{KLD} to the dimension correlation length ξ_δ, strongly suggests that the SDC state is extensively chaotic. We note that the extensive linear scaling of the dimension D_{KLD} with subsystem area is best for smaller subsystems, which we believe is a consequence of the fact that smaller subsystems have a faster time scale to become statistically stationary.

We next compared ξ_{KLD} (computed in the center of the cell for $f = 0.7$) with the two-point correlation length ξ_2 as the reduced Rayleigh number ϵ was varied. The two-point correlation length ξ_2 was calculated from the inverse of the width of the peak in the Fourier spectrum of the spatial data which was pre-multiplied with a hanning window of diameter equal to the lateral dimension of the images. As shown in Table II, the parametric dependences of the length ξ_{KLD} and the two-point correlation length ξ_2 are different [12]. The fact that ξ_{KLD} attains a minimum between $\epsilon = 0.79$ and 0.88, and then increases with increasing ϵ (corresponding to a decrease in complexity) is somewhat counterintuitive since one might have expected ξ_{KLD} to decrease with ϵ as the system is forced further away from equilibrium (more modes are needed to approximate the space-time data). A possible explanation for these opposing trends may be found by examining the different spatial structures in Figs. 1A and 1B. For Fig. 1A ($\epsilon = 0.52$), the area fraction of local straight-roll regions is larger than that for spiral-roll regions, whereas for Fig. 1B ($\epsilon = 0.93$) the relationship between straight- and spiral-roll regions is reversed. We speculate that the data between $\epsilon = 0.79$ and 0.88 consist of nearly equal fractions of straight- and spiral-roll regions and thus require more KLD eigenmodes per unit volume $D_{\mathrm{KLD}}/V_{\mathrm{sub}}$ and so $\xi_{\mathrm{KLD}} = (D_{\mathrm{KLD}}/V_{\mathrm{sub}})^{-1/2}$ is smaller. This speculation about the relative complexity of straight-roll versus spiral-roll regions is supported by examining the KLD spatial eigenmodes for the $\epsilon = 0.52$ and $\epsilon = 0.93$ data. These modes have the same qualitative symmetries up to eigenmode 21 (f=0.45 for $\epsilon = 0.52$), beyond which the local straight-roll regions in the $\epsilon = 0.52$ data entered into the next KLD spatial modes and broadened the KLD eigenvalue

spectrum[2]. In the $\epsilon = 0.93$ data, beyond eigenmode 21 the eigenmodes consisted of lattices of convection rolls.

As the KLD analysis of a subsystem is a local procedure, one can quantify differences in the SDC in different regions of the convection cell and so test the assumption that SDC is dynamically homogeneous. For the $\Gamma = 109$ cell, we investigated a radial dependence of ξ_{KLD} by first establishing that D_{KLD} scaled extensively for an annulus of radius $r_o < r < r_o + d$ and angular sector that varied from $\Delta\theta = 0$ to $\Delta\theta = 2\pi$ radians (extensivity was with respect to $\Delta\theta$). We could not estimate nonuniformity for $r_o < 8d$ because the subsystems were too small, or for $r_o > 45d$ as this exceeded the area imaged by the CCD camera. Fig. 3 shows how the length ξ_{KLD} increases by 10% from $r = 8d$ to $r = 45d$. The variation in the length ξ_{KLD} with radial distance r demonstrates that SDC cannot be considered homogeneous (Fig. 3) even in large aspect ratio cells. Based on this calculation we consider the cell *approximately* uniform for $r < 18d$. The time-average of the SDC data does not resemble the nonuniformity in the KLD length in the interior of the cell, but the more easily computed variance pattern does resemble that of ξ_{KLD}. The 10% nonuniformity of ξ_{KLD} in the large cell may be due to both dynamical and experimental reasons. The dynamical one would be due to the pinning of the pattern by the side-wall which has been shown to cause time-averaged patterns near the sidewall for instance in rotating convection experiments [13]. (We also found nonunformity of ξ_{KLD} near the sidewall in the $\Gamma = 29$ cell.) The experimental source could come from the nonlinearity of shadowgraph method. For thin cells and high epsilon values, the nonlinearity is strong and can make the shadowgraph method sensitive to small optical nonuniformities. Unfortunately, the relative strength of these two sources of nonuniformity cannot be determined.

In summary, we have carried out the first analysis of experimental data using the extensive scaling of KLD in subsystems [2] for the spiral-defect chaos state. KLD analysis is straightforward to apply to small subsystems of different geometry as it does not impose an approximate periodicity of the space-time data as is the case for Fourier analysis. By verifying that the dimension D_{KLD} scaled linearly with subsystem size, we provide strong evidence that the experimental system is extensively chaotic [1]. The utilization of a subsystem was essential for our study for two reasons. First, local analysis of subsystems allowed the characterization of inhomogeneous dynamics in our experimental convection cell. Further, by exploiting the reduced data requirements of subsystems, we could estimate the parametric behavior of the dimension density (the average number of degrees of freedom per unit area) in the center of the convection cell. For the SDC data analyzed, the length ξ_{KLD} seems to quantify differences in the fraction of straight-roll and spiral-defect regions (also observed in the KLD spatial eigenmodes), and so provides information beyond that available from ξ_2. Finally the nonuniformity of the KLD length was shown to extend over 80% of the radius in the $\Gamma = 109$ convection cell. Future analysis will hopefully provide further insights about the transition to SDC and about the chiral-symmetry breaking recently observed in SDC for rotating convection experiments [14].

S.M.Z was supported in part by the Computational Graduate Fellowship Program of the Office of Scientific Computing in the Department of Energy. S.M.Z and H.S.G were supported by NSF grants NSF-DMS-93-07893 and NSF-CDA-92123483-04, and by DOE grant DOE-DE-FG05-94ER25214. J.L, K.B, and G.A were supported by DOE grant DOE-DE-FG03-87ER13738.

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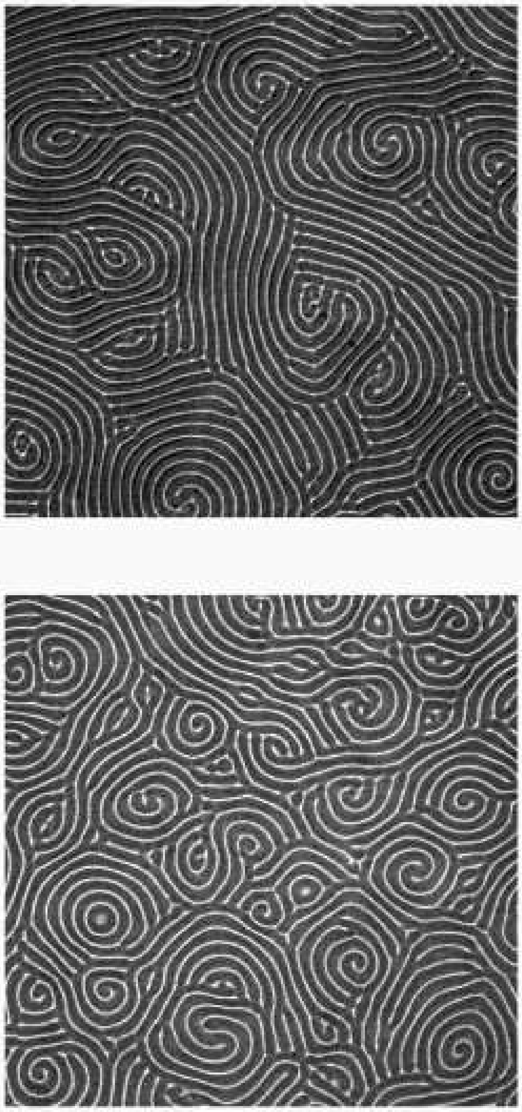
FIGURES

FIG. 1. Time-dependent shadowgraph patterns for the central 43% of the $\Gamma = 109$ convection cell, with light and dark regions corresponding to cool and warm fluid respectively. (A) and (B) are snap-shots of the SDC pattern for reduced Rayleigh number $\epsilon = 0.52$ and $\epsilon = 0.93$ respectively.

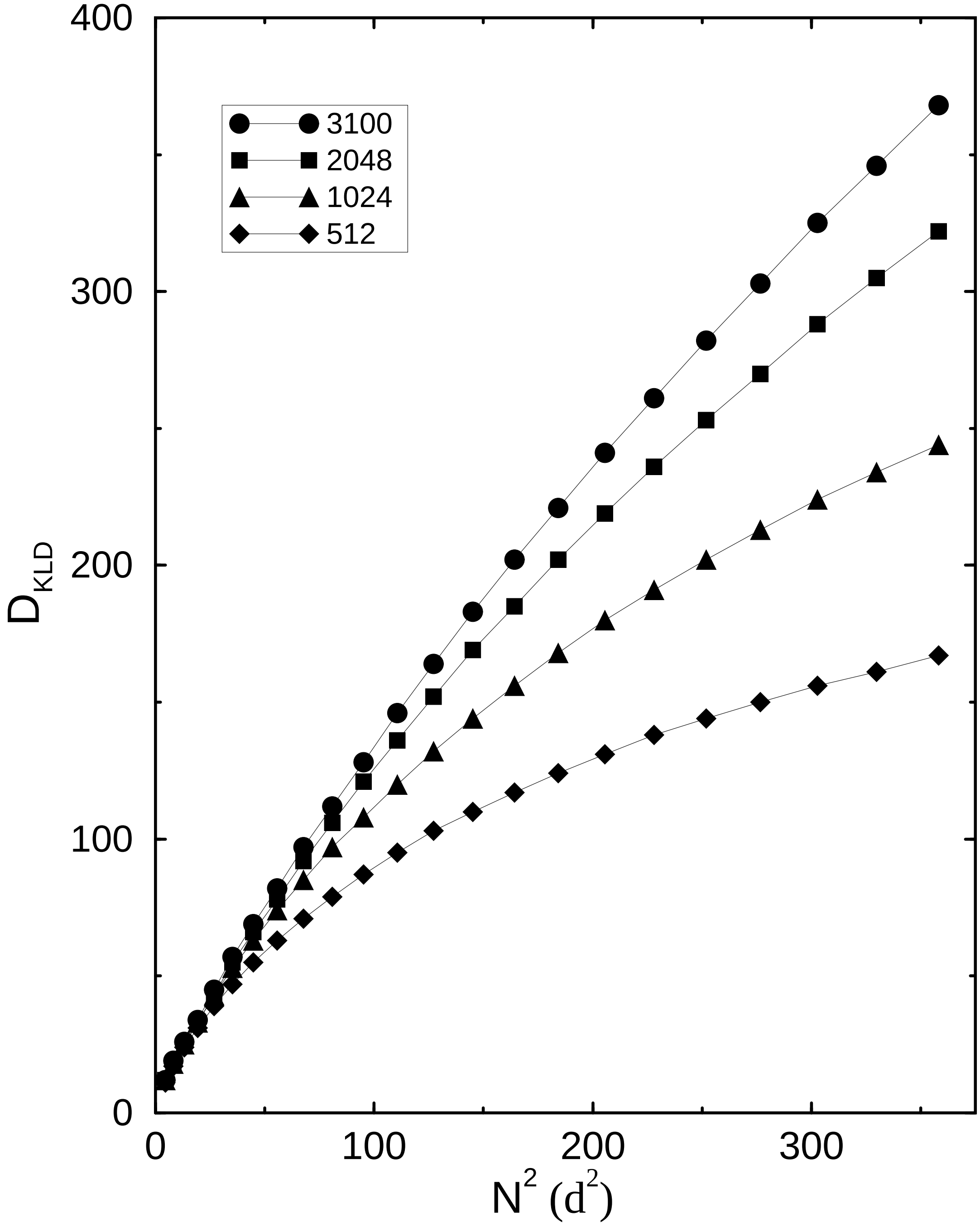
FIG. 2. Scaling of the KLD dimension $D_{\rm KLD}$ with subsystem area N^2 for data in the central 43% of the $\Gamma = 109$ cell with reduced Rayleigh number $\epsilon = 0.52$ and variance fraction $f = 0.70$. The area N^2 of a subsystem is measured in units of d where d is the the thickness of the cell. The labels indicate the number of images used in the calculations and lines connecting points were drawn to guide the eye.

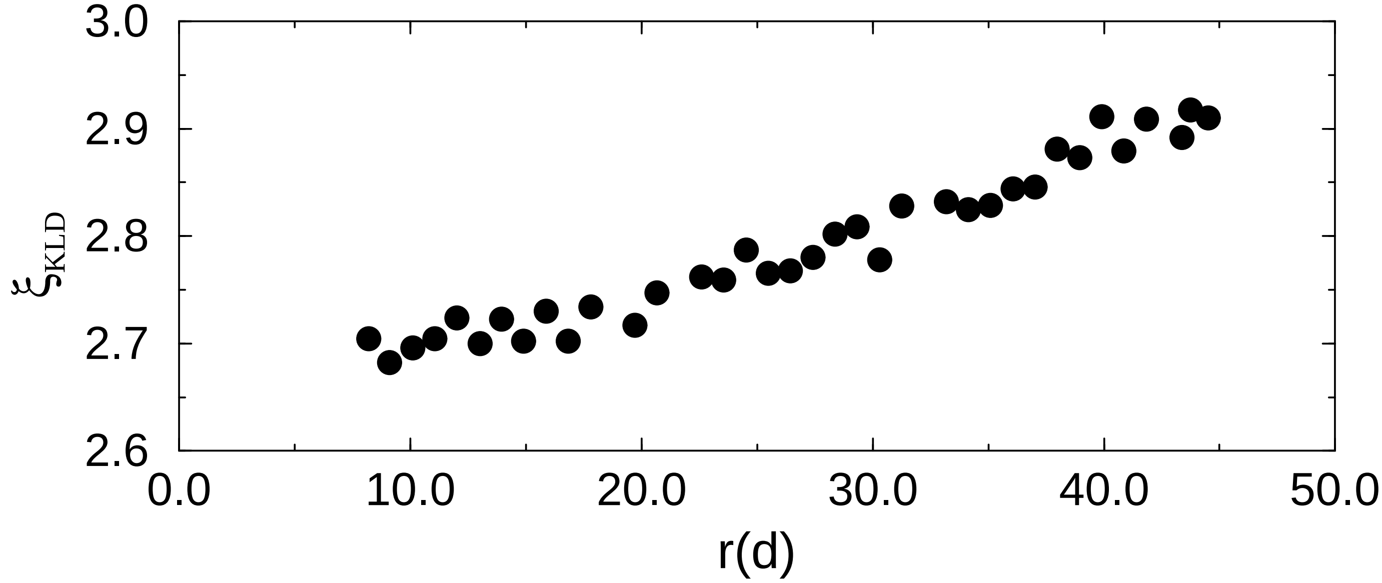
FIG. 3. Radial inhomogeneities of the length $\xi_{\rm KLD}$ in annular subsystems of fixed radii and increasing azimuthal angle for the $\Gamma = 109$ convection cell ($\epsilon = 0.79$). The radial distance r is measured in units of d (the thickness of the convection cell) from the center of the $\Gamma = 109$ convection cell.



A

B


400
300
200
100
0
0
100
200
300
3100
2048
1024
512
D_{KLD}
N^2 (d^2)



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TABLES

TABLE I. Parameters for data used to calculate ξ_{KLD}. Γ is the aspect ratio, ϵ is the reduced Rayleigh number, T is the number of images, and $\Delta t/t_v$ is the sampling rate in units of the vertical thermal diffusion time t_v.

Γ	ϵ	T	$\Delta t/t_v$	t_v (s)
29	1.80	2048	33	8.28
109	0.52	3100	1050	0.86
109	0.67	2500	1050	0.86
109	0.79	2500	1050	0.86
109	0.93	2570	728	0.82

TABLE II. Lengths ξ_{KLD} and ξ_2 (normalized to the depth of the fluid d) as functions of aspect ratio Γ and reduced Rayleigh number ϵ. The length ξ_{KLD} was calculated for fraction $f = 0.7$.

Γ	ϵ	ξ_{KLD}/d	ξ_2/d
29	1.80	1.47	2.28
109	0.52	0.84	4.38
109	0.67	0.81	3.85
109	0.79	0.75	3.60
109	0.88	0.74	3.39
109	0.93	0.90	3.22